Approximate BER Expressions of Distributed Alamouti’s Code in Dissimilar Cooperative Networks with Blind Relays

Zhihang Yi, Student Member, IEEE, and Il-Min Kim, Senior Member, IEEE

Abstract—This paper focuses on error performance analysis of distributed Alamouti’s code. Recently, many works have been devoted to the performance analysis of this code. In order to simplify the analysis, however, they either assumed the channels in the cooperative network had the same variances or only considered asymptotic error performance at high signal-to-noise ratio (SNR) range. In this paper, we study a general dissimilar cooperative network, where the channels have different variances. Two accurate approximate bit error rate (BER) expressions are proposed in order to evaluate the error performance of the distributed Alamouti’s code. We also investigate how the values of channel variances affect the accuracies of the proposed approximate BER expressions. Our results demonstrate that the proposed approximate BER expressions are very close to the exact BER over the whole SNR range. Furthermore, we show that the average BER of the distributed Alamouti’s code behaves like \( \ln(E)/E^2 \) when the transmission power \( E \) is sufficiently large.

Index Terms—Cooperative systems, diversity methods, fading channels, space-time codes.

I. INTRODUCTION

In a cooperative network, several single-antenna relay terminals help the source transmit signals to the destination by forming a distributed multiple-antenna system [1]–[3]. Specifically, in an amplify-and-forward (AF) cooperative network, each relay multiplies the received signal with an amplifying coefficient and then forwards it to the destination. In order to coordinate the transmissions from the relays, repetition-based cooperative strategy was proposed in [3] and its error performance was studied in [4]–[7]. However, the repetition-based cooperative strategy has very poor bandwidth efficiency, because it allows only one relay to forward signals at each time slot. In order to improve the bandwidth efficiency, distributed space-time codes (DSTCs) were proposed and extensively studied [8]–[11].

Many works have been devoted to the error performance and diversity order analysis of DSTCs in AF cooperative networks. In [12], Jing et al. considered a cooperative network with blind relays, i.e., the relays did not have any channel state information (CSI) and set the amplifying coefficients as constants. Furthermore, it was assumed that all the channels in the cooperative network had the same variance. Under this assumption, the authors showed that, at high signal-to-noise ratio (SNR) range, the pairwise error probability of the linear dispersion codes behaves as \( (\ln(E)/E)^K \), where \( E \) is the transmission power and \( K \) is the number of relays. In a cooperative network with blind relays, Anghel et al. [13] considered distributed Alamouti’s code which was single-symbol decodable and achieved the full rate one; but they only found an asymptotic bit error rate (BER) expression of this code. Furthermore, in [14], the authors also considered an Alamouti’s-coded cooperative network with blind relays and they presented a BER expression of the code at high SNR range. At low and moderate SNR range, however, the BER expressions in [13] and [14] were not accurate.

Very recently, Ju et al. found the exact BER expression of the distributed Alamouti’s code [15] in an AF cooperative network with blind relays. However, the authors of [15] assumed that the first-hop channels, i.e., the channels from the source to the relays, had the same variance and the second-hop channels, i.e., the channels from the relays to the destination, also had the same variance. In practice, it is more important to consider a dissimilar cooperative network, where all the channels possibly have different variances. Unfortunately, it is very hard to extend the results in [15] to a dissimilar cooperative network. To the best of our knowledge, the error performance of the distributed Alamouti’s code has not been analyzed in dissimilar cooperative networks with blind relays. This has motivated our work.

In this paper, we study the BER of the distributed Alamouti’s code in a dissimilar AF cooperative network with blind relays. We propose two approximate expressions of the exact BER of the distributed Alamouti’s code. The first approximate BER is very close to the exact BER except when the variances of the second-hop channels are very small. On the other hand, the second approximate BER expression is very accurate except when the variances of the second-hop channels are very large. Therefore, irrespective of the values of the channel variances, we can always use one of the two proposed approximate BER expressions and accurately evaluate the BER of the distributed Alamouti’s code. Furthermore, based on the approximate BER expressions, we show that the average BER of the distributed Alamouti’s code behaves like \( \ln(E)/E^2 \) in a dissimilar AF cooperative network with two blind relays.

The rest of this paper is organized as follows. Section II describes the cooperative network studied in this paper. In Section III, two approximate BER expressions are derived for the distributed Alamouti’s code in a dissimilar cooperative network with blind relays. Then we analyze the diversity order of the code. Section IV presents some numerical results and Section V concludes this paper.
Notations: We use $A = B$ to denote $B$, by definition, equals $A$. For a random variable $X$, $E[X]$ denotes its mean. $X \sim \mathcal{C}(0, \Omega_X)$ means $X$ is a circularly symmetric complex Gaussian random variable with zero mean and variance $\Omega_X$.

II. SYSTEM MODEL

We consider an AF cooperative network with one source, two relays, and one destination. Every terminal has only one antenna and is half-duplex. Let $h_k$ and $f_k$ denote the channel from the source to the $k$-th relay and the channel from the $k$-th relay to the destination, respectively. The channel coefficient $h_k$ is modelled as $h_k = \tilde{h}_k \sqrt{d_{k,d}^{-\beta_{k,d}}}$, where $\tilde{h}_k \sim \mathcal{C}(0,1)$, $\beta_{k,d}$ is the path loss exponent for this channel, and $d_{k,d}$ is the normalized distance from the source to the $k$-th relay. The value of $d_{k,d}$ is decided by $d_{k,d} = \tilde{d}_{k,d}/D$, where $\tilde{d}_{k,d}$ is the actual distance from the source to the $k$-th relay and $D$ is the reference distance determined from measurements. Similarly, we model the channel coefficient $f_k$ as $f_k = \tilde{f}_k \sqrt{d_{k,d}^{-\beta_{k,d}}}$, where $\tilde{f}_k \sim \mathcal{C}(0,1)$, $\beta_{k,d}$ is the path loss exponent for this channel, and $\tilde{f}_k \sim \mathcal{C}(0,1)$, $\beta_{k,d}$ is the path loss exponent for this channel, and $\tilde{f}_k \sim \mathcal{C}(0,1)$. In this paper, we assume all the relays are blind relays as in [4] and [12]-[15], i.e. the relays have no CSI at all. Therefore, the amplifying coefficient $\rho_k$ at the $k$-th relay is given by $\rho_k = \sqrt{E_k/(\Omega_k+\sigma_n^2)}$, where $E_k$ is the transmission power at the source.

At the third and fourth time slots, the two relays use the distributed Alamouti’s code to transmit the signals to the destination. Specifically, at the third time slot, the first relay transmits $\rho_1 y_{1,1}$ and the second relay transmits $-\rho_2 y_{1,2}$ to the destination. Thus, the received signal at the destination is $y_{1,1} = \rho_1 f_1 y_{1,1} - \rho_2 f_2 y_{1,2} + n_{1,1}$, where $n_{1,1}$ is the additive white Gaussian noise and $n_{1,1} \sim \mathcal{C}(0, \sigma_n^2)$. At the fourth time slot, the first relay transmits $\rho_1 y_{1,2}$ and the second relay transmits $\rho_2 y_{1,1}$ to the destination. Consequently, the received signal at the destination is $y_{2,1} = \rho_1 f_1 y_{1,2} + \rho_2 f_2 y_{1,1} + n_{2,1}$, where $n_{2,1}$ is the additive white Gaussian noise and $n_{2,1} \sim \mathcal{C}(0, \sigma_n^2)$. Due to the orthogonal structure of the distributed Alamouti’s code, it is easy to obtain the maximum likelihood (ML) estimate $\hat{x}_1$ of $x_1$ [15] and it is given by $\hat{x}_1 = \rho_1 f_1^* h_{1}^* y_{1} + \rho_2 f_2^* h_{2}^* y_{2}$. Thus, the instantaneous SNR $\gamma$ of $\hat{x}_1$ is given by

$$\gamma = \frac{E_k (\rho_1^2 |f_1|^2 |h_1|^2 + \rho_2^2 |f_2|^2 |h_2|^2)}{\sigma_n^2 (\rho_1^2 |f_1|^2 + \rho_2^2 |f_2|^2 + 1)}.$$  

Similarly, we can obtain the ML estimate $\hat{x}_2$ of $x_2$ and its instantaneous SNR equals to $\gamma$ as well. Furthermore, if $M$-quadrature amplitude modulation (QAM) is used as the modulation scheme, the conditional BER $P_b(\gamma)$, conditioned on instantaneous channel coefficients, of $\hat{x}_1$ or $\hat{x}_2$ is given by [17]

$$P_b(\gamma) = \frac{1}{\log_2 M} \sum_{j=1}^{\log_2 M} P_{b,j}(\gamma)$$

and $P_{b,j}(\gamma)$ is given by

$$P_{b,j}(\gamma) = \frac{2}{\sqrt{M}} \sum_{i=0}^{(1-2^{-i})}\left\{(-1)^{2^{i-1}/M} 2^{j-1} - \frac{2^{j-1} i + 1}{2^{M} \sqrt{M + 1}} \right\}(2i+1)^2 \mathcal{Q}(\frac{3\gamma}{M-1})$$

where $\mathcal{Q}(\cdot)$ denotes the Q-function.

III. BER AND DIVERSITY ORDER ANALYSIS OF THE DISTRIBUTED ALAMOUTI’S CODE

In this section, we consider a dissimilar cooperative network where the variances $\Omega_k$ and $\Omega_f$ of the channels are possibly different. We derive two approximate BER expressions of the distributed Alamouti’s code and analyze the diversity order of this code.

In order to derive the exact BER $P_b$ of $\hat{x}_1$ or $\hat{x}_2$, one needs $\text{MGF}_\gamma(s)$ of $\gamma$, but it is very hard to obtain. Due to this reason, we try to find approximate BER expressions instead.
A. First approximate BER expression

We rewrite the instantaneous SNR $\gamma$ in (1) as

$$\gamma = \frac{E_s}{\sigma_n^2} \left( \min(|h_1|^2, |h_2|^2)(p_1^2|f_1|^2 + p_2^2|f_2|^2) \right) + \frac{p_1^2|f_1|^2 + p_2^2|f_2|^2 + 1}{|f_1|^2 + |f_2|^2 + 1} \left( \max(|h_1|^2, |h_2|^2) - \min(|h_1|^2, |h_2|^2) \right) \times \frac{p_1^2|f_1|^2}{|f_1|^2 + |f_2|^2 + 1} \right) \right)$$

$$\approx: \frac{E_s}{\sigma_n^2} \left( \gamma_{app1}^1 + \gamma_{app1}^2 \right),$$

where $p = \arg\max_{k=1,2} |h_k|^2$. In order to derive an approximate BER expression, it is desirable to find the MGFs of $\gamma_{app1}^1$ and $\gamma_{app1}^2$. To this end, we develop the following lemma.

**Lemma 1:** Assume $X_1$, $X_2$, $Y_1$, and $Y_2$ are random variables with means $a_1$, $a_2$, $b_1$, and $b_2$, respectively. The MGF $MGF_1(s; a_1, a_2, b_1, b_2)$ of the function $\min(X_1, X_2)/(Y_1 + Y_2 + 1)$ is given as follows:

$$MGF_1(s; a_1, a_2, b_1, b_2) = H(s, a_1a_2, a_1 + a_2, b_1, b_2),$$

where the function $H(w, x, y, z)$ is given by

$$H(w, x, y, z) = \frac{1}{(1-wx)^2(y-z)} \left[ (1-wx)(y-z) - wxe^{\pi(-w-\pi)} \frac{1}{y(1-wx)} + wxe^{\pi(-w-\pi)} \frac{1}{z(1-wx)} \right],$$

and $Ei(\cdot)$ is the exponential integral function [18]. Let $p = \arg\max_{k=1,2} X_k$ and then the MGF $MGF_2(s; a_1, a_2, b_1, b_2)$ of the function $(\max(X_1, X_2) - \min(X_1, X_2))/Y_p/(Y_1 + Y_2 + 1)$ is given by

$$MGF_2(s; a_1, a_2, b_1, b_2) = \frac{a_1 + a_2}{(a_1 + a_2)^2} \left[ a_1 + a_2 s M \left( \frac{b_2}{b_1}, -a_1, \frac{1}{b_1} \right) + a_2 + 2a_2 s M \left( \frac{b_2}{b_1}, -a_2, \frac{1}{b_1} \right) \right] + \frac{a_2}{(a_1 + a_2)^2} \left[ a_1 + a_2 s M \left( \frac{b_1}{b_2}, -a_1, \frac{1}{b_2} \right) + a_2 + 2a_2 s M \left( \frac{b_1}{b_2}, -a_2, \frac{1}{b_2} \right) \right]$$

where the function $M(x, y, z)$ is given by

$$M(x, y, z) = \frac{1}{(x-y-1)^2(y+1)} [(y+1)(y+1-x) + e^{\pi}Ei \left( \frac{1}{x} \right) x(y+1) + e^{\pi}Ei \left( \frac{1}{1+y} \right) (xz - (x+z)(y+1))].$$

**Proof:** See Appendix A.

The MGFs $MGF_{\gamma_{app1}}^1(s)$ and $MGF_{\gamma_{app1}}^2(s)$ of $\gamma_{app1}^1$ and $\gamma_{app1}^2$ in (7) can be easily obtained by using (8) and (10), respectively, and they are given by

$$MGF_{\gamma_{app1}^1}(s) = MGF_1 \left( \frac{E_s}{\sigma_n^2}, \Omega_{h1}, \Omega_{h2}, \rho_{1f}^2, \rho_{2f}^2 \right),$$

$$MGF_{\gamma_{app1}^2}(s) = MGF_2 \left( \frac{E_s}{\sigma_n^2}, \Omega_{h1}, \Omega_{h2}, \rho_{1f}^2, \rho_{2f}^2 \right).$$

(12)

(13)

Although $\gamma_{app1}^1$ and $\gamma_{app1}^2$ are dependent with each other, we ignore such dependency in this paper and approximate the MGF $MGF_{\gamma}(s)$ of $\gamma$ in the following way

$$MGF_{\gamma}(s) \approx MGF_{\gamma_{app1}}(s) = MGF_{\gamma_{app1}^1}(s)MGF_{\gamma_{app1}^2}(s).$$

(14)

Based on (14), an approximate BER expression is derived in the following theorem. We will discuss how the approximation in (14), i.e. ignoring the dependency between $\gamma_{app1}^1$ and $\gamma_{app1}^2$, affects the accuracy of the proposed approximate BER expression in detail later.

**Theorem 1:** When $M$-QAM is used as the modulation scheme, the exact BER $P_b$ can be approximated by

$$P_b \approx P_{app1}^b = \frac{1}{\log_2 M} \sum_{j=1}^{\log_2 \sqrt{M}} P_{b,j}^{app1},$$

(15)

where $P_{b,j}^{app1}$ is given by

$$P_{b,j}^{app1} = \frac{2}{\sqrt{M}} \left( 1 - 2^{-j} \right) \sqrt{\frac{1}{M}} \sum_{i=0}^{\left\lfloor \frac{2^{-j-i}}{M} \right\rfloor} \left( -1 \right)^{\left\lfloor 2^{-j-i} \right\rfloor} \left( 2^{-j-1} \sqrt{\frac{1}{M}} + \frac{1}{2} \right) \int_{\theta=0}^{\pi} MGF_{\gamma_{app1}} \left( \frac{3(2i+1)^2}{2(M-1)\sin^2 \theta} \right) d\theta.$$

(16)

**Proof:** The proof is by simply replacing $MGF_{\gamma}(s)$ in (5) by $MGF_{\gamma_{app1}}(s)$.

Although our proposed approximate BER $P_{b_{app1}}$ in (15) contains an integration, it is very easy to calculate because this integration is over a finite range, which has been shown in previous papers [6]. Furthermore, numerical results demonstrate that $P_{b_{app1}}$ is very close to $P_b$ except when the variances $\Omega_{f1}$ and $\Omega_{f2}$ are both very small. This is because the correlation coefficient between $\gamma_{app1}^1$ and $\gamma_{app1}^2$ is very close to zero except that special case, which can be seen from Fig. 4. However, this fact does not imply that $\gamma_{app1}^1$ and $\gamma_{app1}^2$ are nearly independent. They are always dependent on each other and such dependency certainly affects the accuracy of $P_{b_{app1}}$ as we can see in Figs. 1, 3, 5, and 6.

B. Second approximate BER expression

In order to find an accurate approximation of $P_b$ for the case that $\Omega_{f1}$ and $\Omega_{f2}$ are both very small, a second approximate BER expression is derived in the following. We operate the Q-function by $Q(x) \approx e^{-x^2/2} / \sqrt{2} + e^{-x^2/2} / \sqrt{2} \cdot 1 / \sqrt{4} [20]$. By using this approximation, the approximate BER $P_{b_{app2}}$ can be expressed in terms of $MGF_{\gamma_{app2}}(s)$ without any integrations. However, this method may affect the accuracy of $P_{b_{app1}}$, and hence, it is not used in this paper.

3 Actually, a closed-form approximation can be obtained by approximating $Q$-function by $Q(x) \approx e^{-x^2/2} / \sqrt{2} + e^{-x^2/2} / \sqrt{2} \cdot 1 / \sqrt{4} [20]$. By using this approximation, the approximate BER $P_{b_{app2}}$ can be expressed in terms of $MGF_{\gamma_{app2}}(s)$ without any integrations. However, this method may affect the accuracy of $P_{b_{app1}}$, and hence, it is not used in this paper.
instantaneous SNR $\gamma$ in (1) into two terms in a different way as follows:

$$
\gamma = \frac{E_s}{\sigma_n^2} \left( \frac{\rho_1^2|f_1 h_1|^2}{\rho_1^2|f_1|^2 + \rho_2^2|f_2|^2 + 1} \right)
+ \frac{\rho_1^2|f_1|^2 + \rho_2^2|f_2|^2 + 1}{\rho_1^2|f_1 h_1|^2 + \rho_2^2|f_2 h_2|^2 + 1}.
$$

(17)

$$
P_b \approx P_{b, app} = \frac{1}{\log_2 M} \sum_{j=1}^{\log_2 M} P_{b, j, app},
$$

(23)

where $P_{b, j, app}$ is given by

$$
P_{b, j, app} = \frac{2}{\sqrt{M}} \left\{ \left( -1 \right)^{2j-1} \right\}^{-\frac{1}{2}} \left( \frac{1}{2} \right) \int_{0}^{\frac{\pi}{2}} \text{MGF}_{app} \left( -\frac{3(2i+1)^2}{2(M-1)} \right) d\theta
$$

(24)

**Proof:** The proof is by simply replacing MGF$_{\gamma}(s)$ in (5) by MGF$_{app}$. As our first approximate BER expression, the second approximate BER $P_{b, app}$ in (23) also only contains a finite integration and it is very easy to calculate. Moreover, numerical results demonstrate that $P_{b, app}$ is an accurate approximation of $P_b$ when except when $\Omega_1$ and $\Omega_2$ are both very large. This is because the correlation coefficient between $\gamma_{app}$ and $\gamma_{app}$ is very close to zero except that special case as illustrated by Fig. 4. However, this fact does not imply that $\gamma_{app}$ and $\gamma_{app}$ are independent. They are always dependent on each other and this dependency affects the accuracy of $P_{b, app}$ as we can see in Figs. 2, 3, 5, and 6.

Recall that the first approximate BER $P_{b, app}$ is very close to $P_b$ except when $\Omega_1$ and $\Omega_2$ are both very small. Therefore, depending on the values of $\Omega_1$ and $\Omega_2$, one can always use either $P_{b, app}$ or $P_{b, app}$ in order to accurately approximate $P_b$. Specifically, when $\Omega_1$ and $\Omega_2$ are both very small, one should choose $P_{b, app}$; while, when $\Omega_1$ and $\Omega_2$ are both very large, one should use $P_{b, app}$. For the other cases, both $P_{b, app}$ and $P_{b, app}$ can be used, because both of them are very close to $P_b$. The accuracy of $P_{b, app}$ and $P_{b, app}$ will be confirmed by our simulation results in Section IV.

C. Diversity order of the distributed Alamouti’s code in dissimilar cooperative networks

The approximate BER expressions $P_{b, app}$ and $P_{b, app}$ also enable us to find the diversity order of the distributed Alamouti’s code in a dissimilar cooperative network with blind relays. To achieve this, we analyze the asymptotic behavior of $P_{b, app}$ in the following lemma.

**Lemma 3:** In a cooperative network with two blind relays, the exact BER $P_b$ of the distributed Alamouti’s code behaves like $(\ln(E)/E)^2$ when the transmission power $E$ is sufficiently large.

**Proof:** For simplicity, we assume $E_s = E_{r1} = E_{r2} = E$ and $\sigma_n^2 = 1$. Note that this assumption does not change the asymptotic behavior of the average BER as shown in [12]. When $E$ goes to infinity, we have

$$
\lim_{E \rightarrow \infty} \text{MGF}_{app} \left( -\frac{3(2i+1)^2}{2(M-1)} \right) = C \left( \frac{\ln(E)}{E} \right)^2 + \mathcal{O} \left( \frac{1}{E^2} \right),
$$

(25)

where $C$ is a constant. Therefore, when $E$ is large, MGF$_{app} \left( -3(2i+1)^2/(2(M-1)) \right)$ behaves like $(\ln(E)/E)^2$. The same conclusion also applies to MGF$_{app} \left( -2(2i+1)^2/(M-1) \right)$. Thus, when $E$ is large, the approximate BER $P_{b, app}$ behaves like $(\ln(E)/E)^2$. Since $P_{b, app}$ accurately approximate $P_b$, $P_b$ behaves like $(\ln(E)/E)^2$ as well when $E$ is large.

In [12], Jing et al. have shown that the pairwise error probability of the distributed linear dispersion codes behaves like $(\ln(E)/E)^K$ in a cooperative network with $K$ relays. Since distributed Alamouti’s code is a special distributed linear dispersion code, it is not surprising that its error performance behaves like $(\ln(E)/E)^2$. Moreover, when $E$ is sufficiently large, $\ln(E)$ is much smaller than $E$, and hence, $(\ln(E)/E)^2 \approx 1/E^2$. Therefore, we conjecture that

4By following the same approach, we can also show that $P_{b, app}$ behaves like $(\ln(E)/E)^2$ when the transmission power $E$ is sufficiently large.
the distributed Alamouti’s code can achieve the full diversity order two at high SNR range. Note that the result in [12] was based on the assumption that the variances of all the channels were identical. Our work is more general than [12] in the sense that we consider a dissimilar cooperative network where the variances of the channels are possibly different. Note that our analysis is based on the distributed Alamouti’s code which is used in a cooperative network with two relays. When there are more than two relays, it might be hard to extend our analysis to that case.

IV. NUMERICAL RESULTS

In this section, we compare the proposed approximate BER expressions with the exact BER obtained by simulation. We use $M$-QAM as the modulation scheme. The source and the two relays have the same transmission powers, i.e. $E_s = E_{r1} = E_{r2} = E$. Thus, the average SNR per bit equals to $E/(\sigma_n^2 \log_2 M)$. We assume that the source, the relays, and the destination are located in a straight line. Furthermore, we let the reference distance equal to the distance from the source to the destination, and hence, $d_{s,k} = 1 - d_{k,d}$, $k = 1, 2$. We set the path loss exponents as $\beta_{s,k} = \beta_{k,d} = 4$, $k = 1, 2$, in order to model the wireless channels in an urban area. As a result, the channel variances $\Omega_{h_k}$ and $\Omega_{f_k}$ are purely decided by the locations of the relays, i.e. $\Omega_{h_k} = d_{s,k}^{-4}$ and $\Omega_{f_k} = d_{k,d}^{-4}$.

In Fig. 1, we compare the first approximate BER expression $P_b^{\text{app1}}$ with the exact BER $P_b$ which is obtained by averaging the conditional BER $P_b(\gamma)$ in (2) over many channel realizations. In order to test the accuracy of $P_b^{\text{app1}}$, we consider three channel settings by placing the two relays at three different locations. Fig. 1 shows that $P_b^{\text{app1}}$ is very accurate in all channel settings. Furthermore, unlike the BER expressions in [13] and [14], which is accurate only at high SNR range, $P_b^{\text{app1}}$ is accurate over the whole SNR range. In Fig. 2, the second approximate BER expression $P_b^{\text{app2}}$ is compared with $P_b$ under three different channel settings. We see that $P_b^{\text{app2}}$ also provides an accurate approximation of $P_b$ over the whole SNR range.

In Fig. 3, we show how the channel variances affect the accuracies of $P_b^{\text{app1}}$ and $P_b^{\text{app2}}$. We fix the distance between the two relays and move the two relays between the source and the destination in order to change the values of $\Omega_{h_k}$ and $\Omega_{f_k}$. It can be seen that $P_b^{\text{app1}}$ is very accurate when the two relays are close to the destination; while it loses accuracy when the relays move toward the source. This can be explained by Fig. 4 where we investigate the correlation coefficient between $\gamma_1^{\text{app1}}$ and $\gamma_2^{\text{app1}}$ under the same channel setting as in Fig. 3. When the relays get closer to the source, the value of $d_{k,d}$ becomes larger, and hence, the value of $\Omega_{f_k}$ becomes smaller. As a result, the correlation between $\gamma_1^{\text{app1}}$ and $\gamma_2^{\text{app1}}$ gets strong,

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Alamouti’s code and, when the relays are close to the source, or

\[ \gamma \]

where the relays are close to the destination, we should use

\[ d \]

and the value of

\[ \Omega \]

become less accurate. Overall, Fig. 3 demonstrates that, when the relays are close to the destination, the value of

\[ d \]

gets its accuracy.

On the other hand, the second approximate BER

\[ P_{b}^{app2} \]

is about

\[ 0.4 \]

and

\[ 0.45 \]

the difference between

\[ P_{b}^{app2} \]

and

\[ P_{b} \]

about

\[ 0.6 \]

when the average SNR per bit is at

\[ 10 \]

dB. Although the difference between the exact BER and our approximate BER expressions slightly increases with the average SNR per bit, the largest difference is approximately

\[ 1 \]

dB at practical average SNR and BER range. Therefore, our approximate BER expressions still approximate the exact BER with an acceptable accuracy even when the relays are located in the center between the source and the destination.

V. CONCLUSION

The error performance of the distributed Alamouti’s code is studied in this paper. We consider a general dissimilar cooperative network, where the channel variances may have different values. Two approximate BER expressions are derived in order to evaluate the error performance of the distributed Alamouti’s code. The first approximate BER expression is very close to the exact BER except when the variances of the second-hop channels are very small. On the other hand, the second approximate BER expression is very accurate except when the variances of the second-hop channels are very large. Therefore, irrespective of the values of the channel variances, we can always choose one of the two proposed approximate BER expressions and accurately evaluate the error performance of the distributed Alamouti’s code. Based on the approximate BER expressions, we also show that the average BER of the distributed Alamouti’s code behaves like

\[ (\ln(E)/E)^2 \]

in a dissimilar cooperative network with two blind relays.
APPENDIX A

Proof of Lemma 1

We first show the derivation of MGF\(_1(s; a_1, a_2, b_1, b_2)\). Let
\[ Z_1 = \min(X_1, X_2) \text{ and } T_1 = (Y_1 + Y_2)/(Y_1 + Y_2 + 1). \]
It is not hard to find the probability density functions (PDFs) \( f_{Z_1}(z) \) and \( f_{T_1}(t) \) of \( Z_1 \) and \( T_1 \)

\[
f_{Z_1}(z) = \left( \frac{1}{a_1} + \frac{1}{a_2} \right) e^{-\left( \frac{z}{a_1} + \frac{z}{a_2} \right)}, \quad \text{(A.1)}
\]

\[
f_{T_1}(t) = \frac{e^{\frac{-z}{a_1} - \frac{z}{a_2}} - e^{\frac{-z}{b_1} - \frac{z}{b_2}}}{(t-1)^2(b_2 - b_1)}, \quad 0 \leq t \leq 1. \quad \text{(A.2)}
\]

First of all, MGF\(_1(s; a_1, a_2, b_1, b_2)\) can be obtained as follows:

\[
\text{MGF}_1(s; a_1, a_2, b_1, b_2) = \int_0^1 \int_0^\infty e^{sz} f_{Z_1}(z) f_{T_1}(t) dz dt
\]
\[
= \int_0^1 e^{\frac{at}{a_1} + \frac{at}{a_2}} dt \
= H\left(s, \frac{a_1a_2}{a_1 + a_2}, b_1, b_2\right). \quad \text{(A.4)}
\]

The last equation is based on the following equation

\[
\int_0^1 \frac{e^{\frac{z}{a_1} - 1} - e^{\frac{z}{b_1} - 1}}{(t-1)^2(1-wxt)} dt
\]
\[
= \int_0^\infty \frac{e^{\frac{z}{a_1} - \frac{z}{a_2}} - e^{\frac{z}{b_1} - \frac{z}{b_2}}}{(1-wx)u+1} du \quad \text{(A.5)}
\]
\[
= \int_0^\infty \frac{e^{\frac{z}{a_1} - \frac{z}{a_2}} - e^{\frac{z}{b_1} - \frac{z}{b_2}}}{(1-wx)u+1} du \quad \text{(A.6)}
\]
\[
= H(w, x, y, z), \quad \text{(A.7)}
\]

where the integrations are solved by [18, pp. 338, 3.353.5] and [18, pp. 337, 3.352.4].

Secondly, we show the derivation of MGF\(_2(s; a_1, a_2, b_1, b_2)\). Let \( Z_2 = \max(X_1, X_2) - \min(X_1, X_2) \) and \( T_2 = Y_2/(Y_1 + Y_2 + 1) \). The PDF \( f_{Z_2}(z) \) of \( Z_2 \) can be derived from [19]

\[
f_{Z_2}(z) = \frac{e^{-\frac{z}{a_1} + \frac{z}{a_2}}}{a_1 + a_2}. \quad \text{(A.8)}
\]

We assume \( p = 2 \), and hence, \( T_2 = Y_2/(Y_1 + Y_2 + 1) \). It is not hard to find the conditional cumulative distribution function (CDF) \( F_{T_2|p=2}(t) \) of \( T_2 \)

\[
F_{T_2|p=2}(t) = \begin{cases} 
1 - \frac{e^{\frac{-z}{b_1} + \frac{z}{b_2}}}{\frac{z}{a_1} - \frac{z}{a_2}} + 1, & 0 \leq t \leq 1 \\
1, & t \geq 1. \end{cases} \quad \text{(A.9)}
\]

Let \( f_{T_2|p=2}(t) \) denote the PDF of \( T_2 \), then

\[
F_{T_2|p=2}(t) = \frac{1}{a_1 + a_2} \left[ 1 - e^{\frac{-z}{b_1} + \frac{z}{b_2}} \right] + \frac{a_2}{a_1 + a_2} \left[ 1 - e^{\frac{-z}{b_1} + \frac{z}{b_2}} \right]. \quad \text{(A.10)}
\]

Let \( f_{T_2|p=2}(t) \) denote the conditional PDF of \( T_2 \), then the conditional MGF \( \text{MGF}_2(s; a_1, a_2, b_1, b_2|p = 2) \) is given by

\[
\text{MGF}_2(s; a_1, a_2, b_1, b_2|p = 2) = \int_0^1 \int_0^\infty e^{szt} f_{T_2|p=2}(t) dz dt
\]
\[
= \frac{1}{a_1 + a_2} \left[ 1 - e^{\frac{-z}{b_1} + \frac{z}{b_2}} \right] + \frac{a_2}{a_1 + a_2} \left[ 1 - e^{\frac{-z}{b_1} + \frac{z}{b_2}} \right]. \quad \text{(A.12)}
\]

The last equality by using integration by parts and the following equation

\[
\int_0^1 e^{-\frac{z}{b_1} + \frac{z}{b_2}} \frac{1}{(1-y)^2} dt = \int_0^\infty e^{-zw} (y + 1)\left( \frac{(y+1)w+1}{y+1} \right) dy
\]
\[
= M(x, y, z), \quad \text{(A.13)}
\]

where the last step is done by using [18, pp. 337, 3.352.4 and 3.353.3].

Similarly, when \( p = 1 \), the conditional MGF \( \text{MGF}_2(s; a_1, a_2, b_1, b_2|p = 1) \) is given by

\[
\text{MGF}_2(s; a_1, a_2, b_1, b_2|p = 1) = \frac{1}{a_1 + a_2} \left[ 1 - e^{\frac{-z}{b_1} + \frac{z}{b_2}} \right] + \frac{a_2}{a_1 + a_2} \left[ 1 - e^{\frac{-z}{b_1} + \frac{z}{b_2}} \right]. \quad \text{(A.14)}
\]

Based on (A.12) and (A.14), MGF\(_2(s; a_1, a_2, b_1, b_2)\) can be obtained by law of total probability

\[
\text{MGF}_2(s; a_1, a_2, b_1, b_2) = \text{Pr}(p = 1)\text{MGF}_2(s; a_1, a_2, b_1, b_2|p = 1) + \text{Pr}(p = 2)\text{MGF}_2(s; a_1, a_2, b_1, b_2|p = 2). \quad \text{(A.15)}
\]

and hence, it is given by (10).

APPENDIX B

Proof of Lemma 2

Let \( T = Y_1/(Y_1 + Y_2 + 1) \) and the CDF \( F_T(t) \) of \( T \) can be obtained by (A.9). Let \( f_T(t) \) denote the PDF of \( T \), then
MGF$_3(s; a, b_1, b_2)$ is given by

\[
MGF_3(s; a, b_1, b_2) = \int_0^1 \int_0^\infty e^{xst} \frac{1}{a} e^{-x/a} f_T(t) dx dt \tag{D.1}
\]

\[
= \int_0^1 \frac{1}{1 - ast} dF_T(t) \tag{D.2}
\]

\[
= \frac{F_T(t)}{1 - ast} \bigg|_{t=0}^{t=1} - \int_0^1 F_T(t) d \left( \frac{1}{1 - ast} \right) \tag{D.3}
\]

\[
= 1 + asM \left( \frac{b_2}{b_1}, -as, \frac{1}{b_1} \right), \tag{D.4}
\]

where the last step is done by using (A.13).

REFERENCES


