Optimum Beamforming in the Broadcasting Phase of Bidirectional Cooperative Communication with Multiple Decode-and-Forward Relays

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Abstract—This letter focuses on the broadcasting phase of bidirectional cooperative networks with multiple decode-and-forward relays. In this phase, the relays first combine the information-bearing symbols transmitted by the sources, and then broadcast them back to the sources in order to achieve bidirectional communications. Two different combining methods at the relays are considered. The first one is that the relays transmit linear combinations of the information-bearing symbols to the sources by beamforming. We develop an algorithm that can compute the optimum beamforming vector in closed form and this beamforming vector minimizes the outage probability of the bidirectional cooperative network. The second method is that the relays combine the information-bearing symbols by exclusive-or and then transmit them to the sources by beamforming. For this case, we show that the instantaneous signal-to-noise ratios at the sources depend on the values of the information-bearing symbols. Based on [1], the optimum beamforming vector is computed and it minimizes an upper bound of the outage probability of the bidirectional cooperative network.

Index Terms—Cooperative systems, diversity methods, fading channels, beamforming.

I. INTRODUCTION

BiDIRECTIONAL cooperative networks can achieve higher bandwidth efficiency than traditional unidirectional cooperative networks and they have been extensively studied recently [2]–[5]. Information exchange in bidirectional cooperative networks consists of a multiple access (MAC) phase and a broadcasting (BC) phase. In the MAC phase, the sources transmit their information-bearing symbols to the relay(s). In the BC phase, the relay(s) forward their received signal(s) to the sources. However, many previous publications only analyzed a network that has one relay and this relay has one antenna [2]–[5].

In [6] and [7], the authors studied bidirectional cooperative networks where multiple antennas were employed at the single relay and the two sources in the network. On the other hand, in [9]–[15], the authors considered bidirectional cooperative networks with multiple relays. When those relays worked in the amplify-and-forward mode, the authors employed beamforming technique in the BC phase in order to enhance performance [10]–[12]. In particular, the optimum beamforming vectors have been found in [11]. On the other hand, when the relays work in the decode-and-forward mode, the achievable rate region of the system may be limited compared to the amplify-and-forward mode. However, the decode-and-forward mode might be more suitable for digital communications and it has drawn more interests in the study of LTE-Advanced standard. When the relays worked in the decode-and-forward mode, Kim et al. analyzed achievable rate regions and outer bounds of the bidirectional cooperative networks, but beamforming was not used in the BC phase [15]. Very recently, lattice coding in bidirectional cooperative networks was studied in [13] and multihop bidirectional cooperative networks were investigated in [14]. To the best of our knowledge, the use of beamforming in decode-and-forward bidirectional cooperative networks with multiple relays has not been studied before and this has motivated our work.

In this letter, we focus on the BC phase of bidirectional cooperative networks with multiple decode-and-forward relays.1 Since the relays work in the decode-and-forward mode, they can decode the information-bearing symbols from the sources based on the received signals in the MAC phase. In the BC phase, we consider two different methods that the relays can forward those symbols to the sources. The first method is that every relay transmits a linear combination of the information-bearing symbols to all the sources by using beamforming. We develop an algorithm to compute the optimum beamforming vector in closed form and this vector minimizes the outage probability of the bidirectional cooperative networks. The second method is that each relay uses exclusive-or (XOR) to combine the information-bearing symbols and then transmit them by using beamforming. For this case, we also find the optimum beamforming vector based on the method developed in [1].

The rest of this letter is organized as follows. Section II describes the bidirectional cooperative networks considered in this letter. In Section III, every relay transmits a linear combination of the information-bearing symbols and we present an algorithm to compute the optimum beamforming vector. In Section IV, the relays use XOR to combine the information-bearing symbols and we find the optimum beamforming vector based on [1]. Section V shows some numerical results and Section VI concludes this paper.

Notation: bold upper and lower letters denote matrices and vectors. \( I_{N \times N} \) denotes a \( N \times N \) identity matrix and \( \mathbf{0}_{N_1 \times N_2} \) denotes a \( N_1 \times N_2 \) all-zero matrix. \( x(i) \) denotes the \( i \)-th entry of \( x \). \( X(:,i) \) denotes the \( i \)-th column of \( X \). \( X^\dagger \) denotes the conjugate transpose, and \( X^H \) denotes the Hermitian of \( X \), respectively. \( \text{diag}(x_1, \ldots, x_K) \) denotes a \( K \times K \) diagonal ma-

1 It will be interesting to study optimum beamforming by considering the MAC and BC phases jointly. However, this is beyond the scope of this letter.
trix with $x_1, \ldots, x_K$ on its main diagonal, \( \text{diag}[X_1, \ldots, X_N] \) denotes a block-diagonal matrix with $X_1, \ldots, X_N$ on its main diagonal.

II. SYSTEM MODEL

We consider a bidirectional cooperative network with two sources and $K$ decode-and-forward relays. Every terminal has only one antenna and is half-duplex. We use $S_1$, $S_2$, and $R_k$ to denote the first source, the second source, and the $k$-th relay, respectively. We assume block-fading channels in this letter. Specifically, let $h_{k}$ represent the fading coefficient of the channel between $S_1$ and $R_k$, and $f_k$ the channel between $R_k$ and $S_2$. The additive noise associated with every channel is a complex Gaussian random variable with zero mean and unit variance. Source $S_i$, $i = 1, 2$, intends to transmit binary information $s_i = [s_{i,1}, \ldots, s_{i,L}]$ to the other source, where $s_{i,l}$ represents a binary bit. Let $\text{mod}(s_i)$ denote the modulated information-bearing symbol of $s_i$ and it has unit average power. The transmission power at both sources is $E_v$. The total transmission power of all the $K$ relays is $E_r$. Since every terminal is a transmitter and a receiver in bidirectional cooperative networks, we assume that every terminal has full channel state information (CSI) as in [6]–[9], i.e. every terminal knows the values of $h_k$ and $f_k$.

The bidirectional communications between $S_1$ and $S_2$ consist of a MAC phase and a BC phase. Based on the received signals from the two sources in the MAC phase, the relays decode $s_1$ and $s_2$. By using cyclic redundancy codes or other error correcting codes, every relay knows if its decoding of $s_1$ and $s_2$ is correct or not. In the BC phase, the relays that can correctly decode $s_1$ and/or $s_2$ transmit to the two sources simultaneously by using beamforming. This can be done in two different ways as shown in Fig. 1 and described in the following.

Method I: Each relay combines $s_1$ and $s_2$ linearly [7], [8]. If $R_k$ can decode both $s_1$ and $s_2$ correctly, it transmits $v_{k,1}\text{mod}(s_1) + v_{k,2}\text{mod}(s_2)$ to the two sources, where $v_{k,1}$ and $v_{k,2}$ are the beamforming coefficients. If $R_k$ can only decode $s_1$ or $s_2$ correctly, it only transmits $v_{k,1}\text{mod}(s_1)$ or $v_{k,2}\text{mod}(s_2)$ to the two sources. Otherwise, $R_k$ does not transmit any signals in the BC phase. The beamforming coefficients $v_{k,1}$ and $v_{k,2}$ will be discussed in Section III.

Method II: Each relay combines $s_1$ and $s_2$ by using XOR [2], [7], [15]. If $R_k$ can decode both $s_1$ and $s_2$ correctly, it transmits $w_k\text{mod}(s_1 \oplus s_2)$ to the two sources, where $w_k$ is the beamforming coefficient. If $R_k$ can only decode $s_1$ or $s_2$ correctly, it only transmits $w_k\text{mod}(s_1)$ or $w_k\text{mod}(s_2)$ to the two sources. Otherwise, $R_k$ does not transmit any signals in the BC phase. This can be achieved by making every relay broadcast an acknowledgement (ACK) signal once it has successful decoding. Such ACK signaling overhead may be eliminated when the relays work in the amplify-and-forward mode. This is because every relay transmits $s_1$ and $s_2$ to the sources in the BC phase when they work in such mode.

2 The MAC phase is not the focus of this paper. Detailed description of the MAC phase of bidirectional cooperative networks can be found in [2]–[5] and comparisons can be found in [16]. Note that, when physical layer network coding (PNC) is used at the MAC phase, the capacity region of the bidirectional cooperative networks is limited by the MAC capacity region [2]. When time division broadcast (TDBC) is used, however, such limitation does not exist [5]. Moreover, the results derived in this letter can be used in both PNC and TDBC.

3 The sources need to know which relays can correctly decode $s_1$ and/or $s_2$. This can be achieved by having every relay broadcast an acknowledgement (ACK) signal once it has successful decoding. Such ACK signaling overhead may be eliminated when the relays work in the amplify-and-forward mode. This is because every relay transmits $s_1$ and $s_2$ to the sources in the BC phase when they work in such mode.

In this section, we let the relays transmit linear combinations of $s_1$ and $s_2$ to the two sources by using beamforming. We develop an algorithm that can compute the optimum beamforming vector for the relays in closed form.

Without loss of generality, we assume that $R_1, \ldots, R_K$ decode both $s_1$ and $s_2$ correctly; $R_{N_0+1}, \ldots, R_{N_0+N_1}$ only decode $s_1$ correctly; $R_{N_0+N_1+1}, \ldots, R_{N_0+N_1+N_2}$ only decode $s_2$ correctly; the other relays can not correctly decode either $s_1$ or $s_2$. Since each relay combines $s_1$ and $s_2$ linearly, the received signal at $S_1$ is given by

\[
y_{S_1} = \sum_{k=1}^{K} \sqrt{E_{r}} h_k (v_{k,1}\text{mod}(s_1) + v_{k,2}\text{mod}(s_2)) \]

\[
+ \sum_{k=N_0+1}^{N_0+N_1} \sqrt{E_{r}} v_{k,1} h_k \text{mod}(s_1) \\
+ \sum_{k=N_0+N_1+1}^{N_0+N_1+N_2} \sqrt{E_{r}} v_{k,2} h_k \text{mod}(s_2) + n_{S_1},
\]

where $n_{S_1}$ is the additive Gaussian noise at $S_1$. In $y_{S_1}$, $s_2$ is the desired binary information to $S_1$ and $s_1$ is just a self-interference. Since $s_1$, $v_{k,1}$, and $v_{k,2}$ are perfectly known by $S_1$, the terms that contain $\text{mod}(s_1)$ in $y_{S_1}$ can be removed.
and hence, $S_1$ obtains a new signal $\tilde{y}_{S_1}$ as follows:

\[
\tilde{y}_{S_1} = \sum_{k=1}^{N_0} \sqrt{E_k h_k v_{k,2 \mod(s_2)}} + \sum_{k=N_0+1}^{N_0+N_1+1} \sqrt{E_k h_k v_{k,2 \mod(s_2)}} + n_{S_1},
\]

(2)

Based on $\tilde{y}_{S_1}$, $S_1$ decodes $s_2$ and the instantaneous signal-to-noise ratio (SNR) $\gamma_1^L$ at $S_1$ is

\[
\gamma_1^L = E_r v^T h h^T v^*,
\]

(3)

where the vectors $v$ and $h$ are defined as

\[
v = [v_1,2, \ldots, v_{N_0}, v_{N_0+N_1}, v_{N_0+N_1+1}, \ldots, v_{N_0+N_1+2}]^T
\]

(4)

\[
h = [h_1, \ldots, h_{N_0}, h_{N_0+N_1}, \ldots, h_{N_0+N_1+2}, 0_{1 \times (N_0+N_1)}]^T.
\]

(5)

Similarly, the instantaneous SNR $\gamma_2^L$ at $S_2$ is given by

\[
\gamma_2^L = E_r v^T f^T f^H v^*,
\]

(6)

where the vector $f$ is defined as:

\[
f = [0_{1 \times (N_0+N_2)}, f_1, \ldots, f_{N_0}, f_{N_0+1}, \ldots, f_{N_0+N_2}]^T
\]

(7)

The data-rates $R_1^L$ and $R_2^L$ at $S_1$ and $S_2$ are given by $R_1^L = \log_2 \left(1 + \gamma_1^L\right)$ and $R_2^L = \log_2 \left(1 + \gamma_2^L\right)$. Assume that the target rate of the whole bidirectional cooperative network is $R$. Since the two sources in this network are equivalent terminals, it is fair to set the target rate of each source as $R/2$. Furthermore, a bidirectional cooperative network is actually a multiuser system with two users. It is well-known that a multiuser system is in outage when any user is in outage [17]. Therefore, the bidirectional cooperative network is in outage when either $R_1^L$ or $R_2^L$ is smaller than the target rate $R/2$, i.e.

\[
P_{\text{outage}}(R) = \Pr \left( R_1^L < \frac{R}{2} \text{ or } R_2^L < \frac{R}{2} \right)
= \Pr \left( \min(\gamma_1^L, \gamma_2^L) < 2^{R/2} - 1 \right).
\]

(8)

As a result, the outage probability is minimized when the minimum $\min(\gamma_1^L, \gamma_2^L)$ of the instantaneous SNRs are maximized. Since the sources and the relays have full CSI, they can compute the optimum beamforming vector $v_{\text{opt}}$ by solving the following optimization problem

\[
v_{\text{opt}} = \arg \min_{v^H v=1} P_{\text{outage}} = \arg \max_{v^H v=1} \min(\gamma_1^L, \gamma_2^L).
\]

(9)

The constraint $v^H v = 1$ constrains that the total transmission power from the relays is $E_r$.

The minimax optimization problem in (9) can be solved by the algorithm specified by Theorem 1 and Lemma 6 of [1]. However, this algorithm must numerically compute the eigenvalues and eigenvectors of several random matrices in order to solve (9), which may require heavy computational loads. In this letter, we exploit the special structure of $\mathbf{h}$ and $\mathbf{f}$, and we solve (9) in closed form. To this end, we first present the following theorem.

Theorem 1: Assume that $x$, $p$, and $q$ are three $(M_1+M_2) \times 1$ vectors. Assume that $p = [0_{1 \times M_1}, p^T]^T$ and $p_0$ is a $M_2 \times 1$ vector. Assume that $q = [q^*, 0_{1 \times M_2}]^T$ and $q_0$ is a $M_1 \times 1$ vector. Define two matrices $A$ and $B$ as $A = pp^H$ and $B = qq^H$. Then the solution to the following minimax optimization problem

\[
x_{\text{opt}} = \arg \max_{x^H x=1} \min \left( x^T Ax^*, x^T Bx^* \right)
\]

(10)

can be found by the following algorithm:

Step 1: Let $x = V_{ab}^* \mathbf{a}$. The vector $\mathbf{a}$ is given by

\[
\mathbf{a} = \left[ \sqrt{\frac{\lambda_0}{\lambda_a + \lambda_b}}, 0_{1 \times (M_1-1)} \right]^T
\]

(11)

where $\lambda_a = p^H p$ and $\lambda_b = q^H q$. The matrix $V_{ab}$ is defined as

\[
V_{ab} = [V_{ab}^1, V_{ab}^2]_{M \times M},
\]

(12)

The vector $v_m$ can be any vector orthonormal to $p_0/\sqrt{\lambda_a}$ and $v_m$ can be any vector orthonormal to $q_0/\sqrt{\lambda_b}$. Then calculate the value of $\min \left( x^T Ax^*, x^T Bx^* \right)$ and denote its value by $V_1$.

Step 2: Let $x = p/\sqrt{\lambda_a}$ and calculate $\min \left( x^T Ax^*, x^T Bx^* \right)$. Denote its value by $V_2$.

Step 3: Let $x = q/\sqrt{\lambda_b}$ and calculate $\min \left( x^T Ax^*, x^T Bx^* \right)$. Denote its value by $V_3$.

Step 4: If $V_1 = \max(V_1, V_2, V_3)$, $x_{\text{opt}}$ is given by $x_{\text{opt}} = V_{ab}^* \mathbf{a}$; if $V_2 = \max(V_1, V_2, V_3)$, $x_{\text{opt}}$ equals to $p/\sqrt{\lambda_a}$; if $V_3 = \max(V_1, V_2, V_3)$, $x_{\text{opt}}$ equals to $q/\sqrt{\lambda_b}$.

Proof: See Appendix A.

By using the algorithm in Theorem 1, the minimax problem in (9) can be solved in closed form. Specifically, we only need to set $p = f$ and $q = h$, where $\mathbf{h}$ and $\mathbf{f}$ are defined in (5) and (7), respectively. Then the optimum beamforming vector $v_{\text{opt}}$ equals to $x_{\text{opt}}$ that is computed by Theorem 1. As we have stated, the algorithm in [1] can find the optimum beamforming vector as well. However, it needs to numerically compute the eigenvalues and eigenvectors of $A$, $B$, and $A - B$. Since those matrices are random, such computations may involve heavy computational loads. On the other hand, our algorithm in Theorem 1 computes the optimum beamforming vector in closed form, and hence, it is much simpler than the algorithm in [1].

IV. OPTIMUM BEAMFORMING WITH XOR COMBINATION AT THE RELAYS

In this section, the relays first use XOR to combine $s_1$ and $s_2$, and then transmit to the sources by using beamforming. We first show that the instantaneous SNRs at the sources depend on the value of $s_1$ and $s_2$. Then the optimum beamforming vector is computed by using the algorithm in [1].
In this section, we assume that the sources use binary phase shift keying (BPSK) modulation. Therefore, we use \( s_i \) to denote the binary information form \( S_i \). Furthermore, we assume that \( \text{mod}(s_i) = 2s_i - 1 \). Without loss of generality, we assume that \( R_1, \ldots, R_{N_0} \) decode both \( s_1 \) and \( s_2 \) correctly; \( R_{N_0+1}, \ldots, R_{N_0+N_1} \) only decode \( s_1 \) correctly; \( R_{N_0+N_1+1}, \ldots, R_{N_0+N_1+N_2} \) only decode \( s_2 \) correctly; the other relays can not correctly decode either \( s_1 \) or \( s_2 \). Since the relays use XOR to combine \( s_1 \) and \( s_2 \), the received signal at \( S_1 \) is given by

\[
y_{S_1} = \sum_{k=1}^{N_0} \sqrt{E_r} w_k h_{k \text{mod}(s_1 \oplus s_2)} + \sum_{k=N_0+1}^{N_0+N_1+1} \sqrt{E_r} w_k h_{k \text{mod}(s_1)} + \sum_{k=N_0+N_1+2}^{N_0+N_1+N_2} \sqrt{E_r} w_k h_{k \text{mod}(s_2)} + n_{S_1},
\]

Since \( s_1 \) and \( w_k \) are perfectly known at \( S_1 \), the second summation in (15) can be removed and \( S_1 \) obtains a new signal \( \hat{y}_{S_1} \) as follows:

\[
\hat{y}_{S_1} = \sum_{k=1}^{N_0} \sqrt{E_r} w_k h_{k \text{mod}(s_1 \oplus s_2)}
\]

Based on \( \hat{y}_{S_1} \), \( S_1 \) decodes \( s_2 \). On the other hand, \( S_2 \) decodes \( s_1 \) in a similar way. Interestingly, the instantaneous SNRs \( \gamma_i^X \) at \( S_1 \) and \( \gamma_2^X \) at \( S_2 \) depend on the values of \( s_1 \) and \( s_2 \) as shown in the following lemma.

**Lemma 1**: The instantaneous SNR \( \gamma_1^X \) of the received signals at \( S_1 \) is given by

\[
\gamma_1^X = \begin{cases} 
\gamma_{1-a}^X = E_r w^T \hat{h}_1 h_H \hat{h}_1 w^*, & \text{when } s_1 = 0 \\
\gamma_{1-b}^X = E_r w^T \hat{h}_2 h_H \hat{h}_2 w^*, & \text{when } s_1 = 1
\end{cases}
\]

where the vectors \( w, \hat{h}_1, \hat{h}_2 \) are defined as

\[
w = [w_1, \ldots, w_{N_0+N_1+N_2}]^T
\]

\[
\hat{h}_1 = [h_1, \ldots, h_{N_0}, 0_{1 \times N_1}]
\]

\[
\hat{h}_2 = [h_1, \ldots, h_{N_0+N_1+1}, \ldots, h_{N_0+N_1+N_2}]^T
\]

The instantaneous SNR \( \gamma_2^X \) of the received signals at \( S_2 \) is given by

\[
\gamma_2^X = \begin{cases} 
\gamma_{2-a}^X = E_r w^T \tilde{f}_1 f_H \tilde{f}_1 w^*, & \text{when } s_2 = 0 \\
\gamma_{2-b}^X = E_r w^T \tilde{f}_2 f_H \tilde{f}_2 w^*, & \text{when } s_2 = 1
\end{cases}
\]

where the vectors \( \tilde{f}_1, \tilde{f}_2 \) are defined as

\[
\tilde{f}_1 = [f_1, \ldots, f_{N_0}, f_{N_0+1}, \ldots, f_{N_0+N_1}, 0_{1 \times N_2}]^T
\]

\[
\tilde{f}_2 = [f_1, \ldots, f_{N_0}, -f_{N_0+1}, \ldots, -f_{N_0+N_1}, 0_{1 \times N_2}]^T
\]

**Proof:** See Appendix B.

We assume that the values of \( s_1 \) and \( s_2 \) can be 0 and 1 with equal probability. As a result, the outage probability \( P_{X}^{\text{outage}} \) of the bidirectional cooperative network can be defined as

\[
P_{X}^{\text{outage}} = \Pr \left( \frac{1}{2} R_{1-a}^X + \frac{1}{2} R_{1-b}^X < \frac{R}{2} \right)
\]

or

\[
\frac{1}{2} R_{2-a}^X + \frac{1}{2} R_{2-b}^X < \frac{R}{2} \right).
\]

where \( R_{1-a} = \log_2 (1 + \gamma_{1-a}^X) \), \( R_{1-b} = \log_2 (1 + \gamma_{1-b}^X) \), and \( R_{2-a} = \log_2 (1 + \gamma_{2-a}^X) \), \( R_{2-b} = \log_2 (1 + \gamma_{2-b}^X) \).

However, it is very hard to find the optimum beamforming vector \( w_{\text{opt}} \) that minimizes this outage probability. In order to make the optimization problem tractable, we define an upper bound \( P_{X-U}^{\text{outage}} \) of the outage probability \( P_{X}^{\text{outage}} \) as follows:

\[
P_{X-U}^{\text{outage}} = \Pr \left( \min \left( R_{1-a}^X, R_{1-b}^X \right) < \frac{R}{2} \right)
\]

or

\[
\min \left( R_{2-a}^X, R_{2-b}^X \right) < \frac{R}{2} \right).
\]

The optimum beamforming vector \( w_{\text{opt}} \) can be found by minimizing this upper bound

\[
w_{\text{opt}} = \arg \min_{w \in \mathbb{C}^{N}} P_{X-U}^{\text{outage}}
\]

\[
= \arg \max_{w \in \mathbb{C}^{N}} \min \left( \gamma_{1-a}^X, \gamma_{1-b}^X, \gamma_{2-a}^X, \gamma_{2-b}^X \right)
\]

The constraint \( w^H w = 1 \) constrains that the total transmission power from the relays is \( E_r \). Note that every terminal has full CSI and they can compute the values of \( \gamma_{1-a}^X, \gamma_{1-b}^X, \gamma_{2-a}^X, \gamma_{2-b}^X \). Thus, they can compute the optimum beamforming vector. Moreover, by choosing \( w_{\text{opt}} \) based on (27), the upper bound of the outage probability of the bidirectional cooperative network is minimized irrespective of the values of \( s_1 \) and \( s_2 \). The optimization problem in (27) can be solved by the algorithm specified by Theorem 1 and Lemma 6 in [1]. Thus, the optimum beamforming vector \( w_{\text{opt}} \) can be found when the relays combine \( s_1 \) and \( s_2 \) by XOR.

The work in this letter might be extended in two different ways. Firstly, it will be very interesting to study the optimum beamforming vector by jointly considering the MAC and BC phases. Another possible extension is to study the optimum beamforming vector when the two sources have different target rates. This corresponds to more general communication networks. For example, when \( S_1 \) is a basestation and \( S_2 \) is a mobile user, \( S_1 \) usually requires a much higher target rate than \( S_2 \). For these two possible extensions, however, we believe that it will be very hard to find the optimum beamforming vector in closed form as in this letter. Numerical methods might be needed to compute the optimum beamforming vector.

**V. NUMERICAL RESULTS**

In our numerical results, the average SNR is denoted by \( E_r \) and we set \( E_2 = E = E_r = K E \). The two sources and the relays are located in a straight line. The distance between the two sources is set to one. Let \( d_{S_1, R_k} \) denote the distance from \( S_1 \) to \( R_k \) and hence, \( d_{S_1, R_k} = 1 + d_{S_2, R_k} \). Furthermore, the path loss exponents is set to be four.

In Figs. 2 and 3, the relays transmit linear combinations of \( s_1 \) and \( s_2 \) to the sources. The optimum beamforming vector
Fig. 2. Outage probability of a bidirectional cooperative network when the relays combine $s_1$ and $s_2$ linearly, $d_{S_1,R_1} = d_{S_1,R_2} = d_{S_1,R_3} = 0.5$ and $R = 1 \text{ bps/Hz}$. 

Fig. 3. Outage probability of a bidirectional cooperative network when the relays combine $s_1$ and $s_2$ linearly, $d_{S_1,R_1} = d_{S_1,R_2} = d_{S_1,R_3} = 0.5$. 

Fig. 4. Outage probability of a bidirectional cooperative network when the relays combine $s_1$ and $s_2$ by XOR, $d_{S_1,R_1} = d_{S_1,R_2} = d_{S_1,R_3} = 0.5$ and $R = 1 \text{ bps/Hz}$. 

Fig. 5. Outage probability of a bidirectional cooperative network when the relays combine $s_1$ and $s_2$ by XOR, $d_{S_1,R_1} = d_{S_1,R_2} = d_{S_1,R_3} = 0.05$. 

computed by Theorem 1 is employed at the relays. As a performance benchmark, we also consider the networks where the relays adopt equal power allocation. One can see that the outage probability of the bidirectional cooperative network is substantially reduced by using our optimum beamforming vector. Furthermore, it can be shown that the optimum beamforming vector makes the networks achieve the full diversity order $K$ by comparing the outage probability curves with those generated by $1/E^K$.

In Figs. 4 and 5, the relays combine $s_1$ and $s_2$ by XOR. The optimum beamforming vector computed by using [1] is adopted at the relays. We also use the networks with equal power allocation as the performance benchmark. One can see that the optimum beamforming vector can considerably reduce the outage probabilities of the bidirectional cooperative networks. It also makes the networks achieve the full diversity order $K$. Furthermore, by comparing Figs. 2 and 4, we see that, when the relays use XOR combination, the networks have lower outage probabilities than the case that the relays use linear combination. For the latter case, $R_k$ transmits $v_{k,1}\bmod(s_1) + v_{k,2}\bmod(s_2)$. The second term is actually a self-interference to $S_1$, and hence, the power used to transmit this term is wasted. However, when XOR combination is used, $R_k$ transmits $w_{k}\bmod(s_1 \oplus s_2)$ and no transmission power is wasted to transmit self-interferences. Due to this reason, when the relays use XOR combination, the networks have better performance.

VI. CONCLUSION

In this letter, we study the BC phase of bidirectional cooperative networks with multiple decode-and-forward relays. Two different transmission methods at the relays are considered. The first one is that the relays transmit linear combinations
of the information-bearing symbols by beamforming. An algorithm is developed to compute the optimum beamforming vector in closed form. This beamforming vector minimizes the outage probability of the bidirectional cooperative network and makes it achieve the full diversity order. The second method is that the relays combine the information-bearing symbols by XOR. For this case, we show that the instantaneous SNRs at the sources depend on the values of the information-bearing symbols. Based on [1], the optimum beamforming vector is obtained. It minimizes an upper bound of the outage probability of the network and it makes the network achieve the full diversity order as well. Our work might be extended in two ways. The first one is to find the optimum beamforming vector by considering the MAC and BC phases jointly. Furthermore, one can also study the optimum beamforming vector when the two sources have different target rates.

APPENDIX

Appendix A: Proof of Theorem 1

It has been shown by [18, Chapter 2] that the optimum solution $x_{\text{opt}}$ should either maximize $x^T A x^*$, maximize $x^T B x^*$, or make $x^T A x^* = x^T B x^*$. Since $A = pp^H$, it is easy to see that $x^T A x^*$ is maximized when $x = p/\sqrt{\lambda_a}$. Similarly, $x^T B x^*$ is maximized when $x = q/\sqrt{\lambda_b}$. Thus, in order to find the optimum $x$, we only need to find the $x$ that solves the equation $x^T A x^* = x^T B x^*$ or, equivalently, $x^T (A - B) x^* = 0$, under the constraint that $x^T H x^* = 1$. Moreover, if such an $x$ is not unique, we need to choose the one that maximizes $x^T A x^*$ and $x^T B x^*$ at the same time.

Since $A = pp^H$, the matrix $A$ has only one non-zero eigenvalue $\lambda_a = p/\sqrt{\lambda_a}$ and its corresponding eigenvector is $p/\sqrt{\lambda_a}$. Furthermore, let $V_a D_a V_a^H$ denote the eigenvalue decomposition of $A$. It is not hard to show that $D_a = \text{diag}[\lambda_a, 0_{1 \times (M_1 + M_2 - 1)}]$ and

$$V_a = \begin{bmatrix} 0_{M_1 \times 1} & 0_{M_1 \times 1} & \cdots & 0_{M_1 \times 1} & I_{M_1 \times M_1} \\ p_1/\sqrt{\lambda_a} & v_1 & \cdots & v_{M_2-1} & 0_{M_2 \times M_2} \end{bmatrix} =: \begin{bmatrix} 0_{M_1 \times M_2} & I_{M_1 \times M_1} \\ V_a & 0_{M_2 \times M_2} \end{bmatrix},$$

(A.1)

where $p_1$ can be any $M_2 \times 1$ vector orthonormal to $p_1/\sqrt{\lambda_a}$. Similarly, $B = qq^H$, $B$ has only one non-zero eigenvalue $\lambda_b = q/\sqrt{\lambda_b}$ and its corresponding eigenvector is $q/\sqrt{\lambda_b}$. Let $V_b D_b V_b^H$ denote the eigenvalue decomposition of $B$. Then we have $D_b = \text{diag}[\lambda_b, 0_{1 \times (M_1 + M_2 - 1)}]$ and

$$V_b = \begin{bmatrix} q_s/\sqrt{\lambda_b} & u_1 & \cdots & u_{M_1-1} & 0_{M_1 \times M_2} \\ 0_{M_2 \times 1} & 0_{M_2 \times 1} & \cdots & 0_{M_2 \times 1} & I_{M_2 \times M_2} \end{bmatrix} =: \begin{bmatrix} V_b & 0_{M_2 \times M_1} \\ 0_{M_2 \times M_1} & I_{M_2 \times M_2} \end{bmatrix},$$

(A.2)

where $u_1$ can be any $M_1 \times 1$ vector orthonormal to $q_s/\sqrt{\lambda_b}$.

By the definition of $A$ and $B$, the matrix $A - B$ equals to $\text{diag}[p_s, q_s^H]$. It is known that $A - B$ has two non-zero eigenvalues and they are given by $\lambda_{a1}$ and $-\lambda_b$. Let $V_{ab} D_{ab} V_{ab}^H$ denote the eigenvalue decomposition of $A - B$. Then $D_{ab}$ can be expressed as $D_{ab} = \text{diag}[\lambda_1, 0_{1 \times (M_2 - 1)}, -\lambda_b, 0_{1 \times (M_1 - 1)}]$ and $V_{ab}$ can be expressed by (12).

We define a new vector $\bar{x}$ as $\bar{x} = V_{ab}^T x$. Since $V_{ab}$ is a unitary matrix, we have $x^T H x^* = \bar{x}^T H \bar{x} = 1$. Thus, the value of $x^T (A - B) x^*$ is given by

$$x^T (A - B) x^* = \bar{x} D_{ab} \bar{x}^H = \lambda_a |\bar{x}(1)|^2 - \lambda_b |\bar{x}(M_2 + 1)|^2.$$

(A.3)

Therefore, in order to make $x^T (A - B) x^* = 0$, $\bar{x}(1)$ and $\bar{x}(M_2 + 1)$ needs to satisfy the following equation

$$\lambda_a |\bar{x}(1)|^2 - \lambda_b |\bar{x}(M_2 + 1)|^2 = 0.$$

(A.4)

We assume that $|\bar{x}(1)|^2 + |\bar{x}(M_2 + 1)|^2 = c$, where $0 \leq c \leq 1$ is a constant and it will be decided later. Based on this assumption and (A.4), the values of $|\bar{x}(1)|$ and $|\bar{x}(M_2 + 1)|$ are given by

$$|\bar{x}(1)| = \sqrt{\frac{c \lambda_b}{\lambda_a + \lambda_b}}, \quad |\bar{x}(M_2 + 1)| = \sqrt{\frac{c \lambda_a}{\lambda_a + \lambda_b}}.$$ (A.5)

In the following, we decide the value of $c$ in order to maximize $x^T A x^*$ and $x^T B x^*$. To this end, we define a new matrix $V$ as $V = V_a^H V_b$. Since $V_a$ and $V_b$ are given by (A.1) and (12), it is easy to see that $V = \text{diag}[I_{M_2 \times M_2}, V_{ab}]$. Then, the value of $x^T A x^*$ is given by

$$x^T A x^* = \bar{x}^T V D_a V_a^H \bar{x}^* = \lambda_a \bar{x}^T V (:,1) V(:,1)^H \bar{x}^* = \lambda_a |\bar{x}(1)|^2 = \frac{c \lambda_a \lambda_b}{\lambda_a + \lambda_b}.$$ (A.6)

Based on (A.6), we should let $c = 1$ in order to maximize $x^T A x^*$. Note that the phases of $\bar{x}(1)$ and $\bar{x}(M_2 + 1)$ do not affect the value of $x^T A x^*$, and hence, can be set arbitrarily. In all, when $\bar{x}$ is given by (11) and $x = V_{ab}^T \bar{x}$, we have $x^T A x^* = x^T B x^*$ and $x^T H x^* = 1$. Moreover, the values of $x^T A x^*$ and $x^T B x^*$ are maximized simultaneously.

Appendix B: Proof of Lemma 1

We first consider the value of $\gamma_1^X$. Based on $\hat{y}_{s_1}$ in (16), the maximum likelihood estimation $\hat{s}_2$ of $s_2$ is performed as follows:

$$\hat{s}_2 = \text{arg} \min_{s_2 \in \{0,1\}} |\hat{y}_{s_1} - \sqrt{E_r} \left( \sum_{k=1}^{N_0} w_k h_k \text{mod}(s_1 + s_2) + \sum_{k=N_0+N_1+N_2}^{N_0+N_1+N_2} w_k h_k \text{mod}(s_2) \right) |^2.$$ (B.1)

When $s_1 = 0$ and $s_2 = 0$, we have $\text{mod}(s_1 + s_2) = -1$ and $\text{mod}(s_2) = 1$. Thus, by following the approach in [19, Chapter 5], it is not hard to show that the conditional error probability $P_1$ conditioned on channel coefficients, is given by

$$P_1 = Q \left( \sqrt{2 E_r E_r - \sum_{k=1}^{N_0} w_k h_k - \sum_{k=N_0+N_1+N_2}^{N_0+N_1+N_2} w_k h_k} \right)^2.$$ (B.2)

Based on (B.2), it is easy to see that the instantaneous SNR $\gamma_1^X$ equals to $E_r w_1^H h_1 w^*$ when $s_1 = 0$ and $s_2 = 0$. For
another case that \( s_1 = 0 \) and \( s_2 = 1 \), we have \( \text{mod}(s_1 \oplus s_2) = 1 \) and \( \text{mod}(s_2) = 1 \). It can be shown that the conditional error probability \( P_2 \) is given by (B.2) as well, and hence, the instantaneous SNR \( \gamma_1^X \) still equals to \( E_r w^T \hat{h}_1 h_1^H w^* \).

When \( s_1 = 1 \) and \( s_2 = 0 \), however, the instantaneous SNR \( \gamma_1^X \) has a different expression. For this case, the conditional error probability \( P_2 \), conditioned on channel coefficients, becomes

\[
P_2 = Q\left( \frac{2E_r}{\sum_{k=1}^{N_0} w_k \hat{h}_k - \sum_{k=N_0+N_1+1}^{N_0+N_1+N_2} w_k \hat{h}_k} \right)^2 \quad (B.3)
\]

Thus, the instantaneous SNR \( \gamma_1^X \) equals to \( E_r w^T \hat{h}_2 \hat{h}_2^H w^* \) when \( s_1 = 1 \) and \( s_2 = 0 \). When \( s_1 = 1 \) and \( s_2 = 1 \), the instantaneous SNR \( \gamma_1^X \) equals to \( E_r w^T \hat{h}_3 \hat{h}_2^H w^* \) as well. In all, the instantaneous SNR \( \gamma_1^X \) depends on the value of \( s_1 \) and its expressions are given by (17). By following the same approach, we can show that the instantaneous SNR \( \gamma_2^X \) at \( S_2 \) depends on the value of \( s_2 \) and its expressions are given by (21).

### References


